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Note

Full Müntz Theorem in $L_p[0, 1]$

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The theorem characterizes sequences $\{\lambda_i\}_0^\infty$ for which the Müntz space $\text{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}$ is dense in $L_p[0, 1]$, $1 < p < \infty$.

We prove the following conjecture of Borwein and Erdélyi [1]:

THEOREM 1 (Full Müntz Theorem in $L_p[0, 1]$. *Let* $1 < p < \infty$ *and* $\{\lambda_i\}_0^\infty$ *be a sequence of distinct real numbers greater than* $-1/p$. *Then*[1]

$$\text{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}$$

is dense in $L_p[0, 1]$ *if and only if*

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \infty. \tag{1}$$

This theorem is a counterpart of the classical Müntz theorem in the uniform metric [2]. As it has been proved by P. Borwein and T. Erdélyi [1], the theorem holds when $p = 1$, and also when $p = \infty$ and $\text{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}$ is replaced by $\text{span}\{1, x^{\lambda_0}, x^{\lambda_1}, ...\}$. The case $p = 2$ of the theorem was obtained by Szász [4] and forms the foundation for the proofs in [1] when $p = 1$ and $p = \infty$. We provide the proof in the case $1 < p < \infty$ and refer the reader to [1] for a detailed discussion and related bibliography.

[1] $\text{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}$ denotes the collection of finite linear combinations of functions $x^{\lambda_0}, x^{\lambda_1}, ...$ with real coefficients.

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Proof of Theorem 1. We will use the following lemma:

LEMMA 1. *Suppose* $\{\mu_i\}_0^\infty$ *is a sequence of distinct positive real numbers such that* $\operatorname{span}\{x^{\mu_i - 1/r}\}_0^\infty$ *is dense in* $L_r[0, 1]$. *Then* $\operatorname{span}\{x^{\mu_i - 1/s}\}_0^\infty$ *is dense in* $L_s[0, 1]$ *for every* $s > r$, *and* $\operatorname{span}\{1, x^{\mu_0}, x^{\mu_1}, ...\}$ *is dense in* $C[0, 1]$.

Proof. Let J be a bounded linear operator from a normed space X into a normed space Y such that $J(X)$ is dense in Y. If $A \subset X$ is dense in X, then $J(A)$ is dense in Y. Let $X = L_r[0, 1]$, $Y = L_s[0, 1]$, $1 \leqslant r < s < \infty$, $A = \operatorname{span}\{x^{\mu_i - 1/r}\}_0^\infty$, and

$$(J\varphi)(x) = x^{-(1/r' + 1/s)} \int_0^x \varphi(t)\, dt, \qquad \frac{1}{r} + \frac{1}{r'} = 1.$$

The boundedness of J: $L_r[0, 1] \to L_s[0, 1]$ follows from the appropriate Hardy-type inequality (see, e.g., [3, Theorem 5.9]). We have $J(x^\lambda) = (\lambda + 1)^{-1} x^{\lambda + 1/r - 1/s}$ when $\lambda > -1$, and $(J\psi_n)(x) = x^n$ when $\psi_n(x) = (n + 1/r' + 1/s)\, x^{n + 1/s - 1/r}$, $n = 0, 1, 2, ...$; by the Weierstrass Approximation Theorem, $J(X)$ is dense in Y. Consequently, $J(A) = \operatorname{span}\{x^{\mu_i - 1/s}\}_0^\infty$ is dense in $Y = L_s[0, 1]$ whenever $A = \operatorname{span}\{x^{\mu_i - 1/r}\}_0^\infty$ is dense in $L_r[0, 1]$. A similar argument with the operator $(J\varphi)(x) = x^{-1/r'} \int_0^x \varphi(t)\, dt$ when $x \in (0, 1]$, $(J\varphi)(0) = 0$, implies that $\operatorname{span}\{1, x^{\mu_0}, x^{\mu_1}, ...\}$ is dense in $C[0, 1]$. ∎

In [1], the theorem was proved when $p = 1$ (the Full Müntz Theorem in $L_1[0, 1]$), and also when $p = \infty$ and $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}$ is replaced with $\operatorname{span}\{1, x^{\lambda_0}, x^{\lambda_1}, ...\}$, $\lambda_i > 0$ (the Full Müntz Theorem in $C[0, 1]$). We use Lemma 1 to extend these results to the case $1 < p < \infty$.

Let $1 < p < \infty$ and $\{\lambda_i\}_0^\infty$ be a sequence of distinct real numbers greater than $-1/p$ and satisfying the condition (1). Then $\{\nu_i = \lambda_i - 1/p'\}_0^\infty$, $1/p + 1/p' = 1$, is a sequence of real numbers greater than -1 and satisfying the condition

$$\sum_{i=0}^{\infty} \frac{\nu_i + 1}{(\nu_i + 1)^2 + 1} = \infty.$$

By the Full Müntz Theorem in $L_1[0, 1]$ ([1, Theorem 2.3]), the set $\operatorname{span}\{x^{\nu_i}\}_0^\infty = \operatorname{span}\{x^{\mu_i - 1}\}_0^\infty$, $\mu_i = \nu_i + 1 = \lambda_i + 1/p$, is dense in $L_1[0, 1]$. Lemma 1 with $r = 1 < p = s$ implies that $\operatorname{span}\{x^{\mu_i - 1/p}\}_0^\infty = \operatorname{span}\{x^{\lambda_i}\}_0^\infty$ is dense in $L_p[0, 1]$.

Conversely, let $1 < p < \infty$ and $\{\lambda_i\}_0^\infty$ be a sequence of distinct real numbers greater than $-1/p$ such that $\operatorname{span}\{x^{\lambda_i}\}_0^\infty$ is dense in $L_p[0, 1]$. Denote $\mu_i = \lambda_i + 1/p$. Then $\operatorname{span}\{x^{\mu_i - 1/p}\}_0^\infty = \operatorname{span}\{x^{\lambda_i}\}_0^\infty$ is dense in

$L_p[0,1]$, and, by Lemma 1, $\operatorname{span}\{1, x^{\mu_i}\}_0^\infty$ is dense in $C[0,1]$. Using the Full Müntz Theorem in $C[0,1]$ ([1, Theorem 2.1]), we obtain the inequality

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \sum_{i=0}^{\infty} \frac{\mu_i}{\mu_i^2 + 1} = \infty.$$

The theorem is proved. ∎

Acknowledgments

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